

# Quasiperiodic Bloch-like states in a surface wave experiment

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Bloch-like surface waves associated with a quasiperiodic structure are observed in a classic wave propagation experiment which consists of pulse propagation with a shallow fluid covering a quasiperiodically drilled bottom. We show that a transversal pulse propagates as a plane wave with quasiperiodic modulation, displaying the characteristic undulatory propagation in this quasiperiodic system and reinforcing the idea that analogous concepts to Bloch functions can be applied to quasicrystals under certain circumstances.

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The main difficulty towards the development of a systematic analytic approach to the transport properties of quasiperiodic systems has been the absence of an analogous Bloch theorem approach as used in the periodic case. In the first efforts to apply a modified version of the Bloch theorem, it was noticed that the dense spectrum of quasiperiodic systems is dominated by only a few special reciprocal lattice points that may be taken to construct a quasi-Brillouin zone [1]. Thus, by considering only the dominant Fourier components, the atomic distribution can be expanded in terms of a discrete aperiodic lattice. Wave functions of the form  $\Psi_{\mathbf{k}} = u_{\mathbf{k}}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$  will therefore solve the Schrodinger equation. In this case  $u_{\mathbf{k}}$  is quasiperiodic and should formally be defined on a countable dense set of reciprocal lattice vectors. But, by the above considerations, this expansion is useful since the Fourier development of the modulation function  $u_{\mathbf{k}}$  can be restricted to the few special reciprocal vectors that dominate the spectra. Thus, Bloch-like states could describe the plane wave propagation in so schematized quasicrystals and free-electron-like bands are expected. Recently this idea was experimentally tested showing that analogous concepts to Bloch functions can be applied to quasicrystals [2].

The classic wave propagation in quasicrystalline systems was addressed in a first seminal acoustic experiment of He and Maynard [3] by the feature that acoustical waves are ideal tools to investigate formally similar

quantum propagation effects [4]. On the other hand, appearance of the quasicrystalline symmetry in fluids dynamics was firstly predicted theoretically by Zaslavsky and co-workers [5] and a simulation similar to the conditions of the present experiments was reported in Ref. [6]. Finally, compressible quasisymmetric flows were considered in Ref. [7], whereas a general outlook on order and disorder in fluid motion can be found in the experiments of Gollub [8].

In this letter we shall see that a discrete restricted spectral scenario can be displayed by means of impulsive waves in hydrodynamic quasicrystals, where we observe Bloch-like surface waves. The waves are generated at the frequencies corresponding to the Fourier components of the quasiperiodic structure at the dominant diffraction spots. The observed Bloch-like waves are plane waves with quasiperiodic modulation generated when a pulse propagates transversally to the quasiperiodic structure. Liquid surface waves shape a quasiperiodic grid that obeys the so-called Octonacci sequence, previously studied [9, 10] but never observed in any experiment.

The quasiperiodic structure involved in our experiment is the octagonal Ammann-Beenker tiling composed by squares and rhombuses [11]. Associated with this octagonal tiling is a quasiperiodic sequence, named the Octonacci sequence [9], that can be generated starting from two steps  $L$  and  $S$ , which are related according to the irrational ratio  $L/S = 1 + \sqrt{2}$ , by iteration of substitution rules:  $L \rightarrow LSL$  and  $S \rightarrow L$ .

The experiments are performed with surface waves generated on a shallow fluid that covers the quasiperiodically drilled bottom of a transparent vessel. Such experiments are similar to others realized in vessels with

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periodic bathymetry and described elsewhere [12, 13] but here not only a continuous wave excitation driven by a vertical monofrequency vibration but also a new experiment of transversal pulse propagation is performed. The bottom dimples are located at 121 vertices of the octagonal tiling. The edge length  $l$  of the tiling is 8 mm with an error lower than 0.4%, the radius  $r$  of the cylindrical bottom wells is 1.75 mm with an error lower than 1% and their depth  $d$  is 2 mm. The depth of the liquid layer over the cylindrical wells is given by  $h_2 = h_1 + d$ , where  $h_1$  is the depth of the thin liquid layer covering the bottom of the vessel among holes.

Under conditions of continuous-wave excitation, an inertial hydrodynamical undulatory instability grows over the bottom wells when the system vibrates vertically at a frequency of 35 Hz. Such an instability becomes remarkable (Fig. 1) due to the high density and the very low surface tension of the liquid [14]. Oscillating bulges over dimples are connected by surface waves with shorter wavelength that decorate the shallow liquid region among holes with  $h_1$  being 0.4 mm. This is a physical scenario similar to that of the Kronig-Penney model but adapted here to a 2D quasiperiodic system. It should be remarked that Fig. 1 is the first available experimental example of a quasiperiodic pattern of waves not arising from a non-linear instability such as the Faraday instability. Standing waves are coupled to only two rings of Fourier wave components of the quasiperiodic bottom structure, as shown in the inset of Fig. 1. Their wavenumber ratio is  $1 + \sqrt{2}$ , as it can be measured in the diffraction pattern (see below). If the effective Fourier transform of a quasiperiodic structure is restricted to a discrete set of Fourier peaks as in this case, then Bloch-like modulation functions can be used to describe the wave propagation in such a simplified quasicrystal [15]. The pattern of Fig. 1 is due to the strong coupling between liquid surface waves and the bottom quasiperiodic topography and does not depend on the shape of the vessel boundary. Undistinguishable patterns are generated with octagonal or circular boundaries. Although the boundaries are reflecting vertical walls, the boundary symmetry matches the symmetry of this standing wave experiment allowing the mentioned boundary-independent strong wave coupling. Due to the accuracy on realizing the setup, localization phenomena do not appear in the described experiment. However, slight tiltings of the vessel generate wave domains [12, 13]; furthermore, point and linear defects can be easily introduced in the system by dropping mercury on the bottom dimples to study new interesting wave localization phenomena [16].

The propagation of a plane wave through the octagonal quasiperiodic structure can be visualized by means of an experiment of wave pulse propagation. A coupling between the vertical waves of the vessel-liquid system and the surface waves generated by a transversal pulse is expected, modelling in this way the propagation of a plane wave through the quasiperiodic structure. This experiment was realized in a vessel with octagonal boundary.

The octagon side  $L$  is 4 cm and it is perpendicular to the  $\Gamma - X$  direction of the well structure. The surface ratio  $f$  between bottom holes and the whole octagon is about 0.15 with  $h_1$  being 0.5 mm. The system is excited near the octagonal boundary with a wave pulse parallelly to the liquid surface and perpendicularly to a side of the octagon. The signal is picked up by means of a Brüel & Kjaer 4344 accelerometer placed at the center of the vessel and it is processed by means of a digital acquisition system. The impulsive signal and the corresponding Fourier transform are shown in Fig. 2 (inset and solid line, respectively). Three clear spectral peaks of the vibrational vessel-liquid system appear at about 20, 30 and 50 Hz. As we shall see, such resonances indicate the existence of three narrow band gaps in the liquid surface wave propagation [13], *i.e.* standing liquid waves are generated at approximately the above generated frequencies.

At the start of each pulse the liquid *feels* the perturbation and a nice quasicrystalline surface wave pattern suddenly appears [Fig. 3(a)]. A transitory weak turbulence arises in the system after scarcely 0.04 s [Fig. 3(b)], whereas robust standing waves drawing clear quasiperiodic grids can be observed between 0.08 and 0.24 s on the liquid surface [Fig. 3(c)]. Finally, quasiperiodic grid patterns decay until the arrival of the next pulse. As wave phase velocities are about 11 cm s<sup>-1</sup> and the wave group velocity is nearly null near the gaps, times for an echo at the boundaries to come back are much longer than observation time.

The robust quasiperiodically spaced standing waves shown in [Fig. 3(c)] are generated by discrete Bragg resonances and thus can be considered quasiperiodic Bloch-like waves. To verify this, first note that the irrational ratio  $LS/L = \sqrt{2}$  is apparent in our experiment [Fig. 3(c)]. Now, using a crystallography-oriented computer program [17], the Fourier transform of the pattern of Fig. 3(c) is calculated and shown in Fig. 4 (top left). Such diffraction pattern matches with an adequate subset of the diffraction pattern of the direct product of both orthogonal Octonacci sequences calculated according to theoretical methods [9, 18] as shown in Fig. 4 (top right). The absence of some diffraction peaks indicates the directional character of the impulsive action. The pulse runs along the  $\Gamma - X$  direction from the upper left to the bottom right corner in both patterns at the top of Fig. 4. Along this direction, an intensity profile is taken in the experimental pattern and recovered the inverse Fourier transform of that unidimensional diffraction subset. The result is shown in Fig. 4 (bottom) which displays an Octonacci sequence, and it matches with that generated theoretically starting from the above mentioned substitution rules. Finally, the above described intensity profile along the  $\Gamma - X$  direction is scaled according to the wavenumber of the waves of the diffraction pattern shown in Fig. 3(c). Such scale is then changed according to the approximate dispersion relationship given [13] by

$$\omega^2 = gk \left( 1 + \frac{T}{\rho g} k^2 \right) \tanh(kh_0),$$

where  $h_0 = h_1(1 - f) + h_2f$ ,  $\omega$  is the angular frequency,  $k$  is the wave number,  $g$  is the acceleration due to gravity,  $T$  is the liquid surface tension and  $\rho$  is the liquid density [14]. In Fig. 2 (dashed lines) the gray scale intensity profile is plotted versus the frequency according to the above mentioned change of scale. The first maximum is scaled by the wavenumber  $k = 11 \text{ cm}^{-1}$ , that corresponds to the main ubiquitous wave appearing in the experimental pattern of Fig. 3(c). As it can be seen, diffraction peaks which represent narrow band gaps closely match in frequency with those independently measured also in Fig. 2. Thus, in this restricted scenario, where the resonances of the vibrational coupling generates a discrete spectrum, the wave pattern observed in Fig. 3(c) corresponds to quasiperiodic Bloch-like states.

If experiments of pulse propagation are realized in vessels with periodic bathymetry [12, 13] no signal of turbulence appears. Thus, as remarked in a different context [10, 19], the quasiperiodicity of the hydrodynamical system could be the origin of the weak chaos observed in the described experiments just at the start of pulses, when amplitudes are higher and hence the non-linearity is stronger. Then, a rapidly increasing number of incommensurable Fourier harmonics can grow due to the finite frequency bandwidth of the pulse and the incommensurate nature of the system. This gives rise to the pre-turbulent state of the surface waves. When the

multiscattering becomes weaker, the Fourier mode cascade decays and the propagative wave exhibits a clean quasiperiodic grid pattern.

Anyway, the quasiperiodic structure underlies in spite of the weak turbulence apparent in Fig. 3(b), which must be looked at grazing incidence to recognize Octonacci quasiperiodic sequences. This is evident in Fig. 5, which is the inverse Fourier transform of Fig. 3(b). The direct product of two orthogonal Octonacci sequences is recovered there, showing a patch of the well known octagonal tiling filled with square and rhombic tiles [11].

In conclusion, we have shown Bloch-like surface waves associated with a quasiperiodic structure in a classic wave propagation experiment. These waves draw clear quasiperiodic grids that obey the Octonacci sequence. Our results along with earlier ones [3] can be helpful to understand the characteristic undulatory propagation in quasiperiodic systems.

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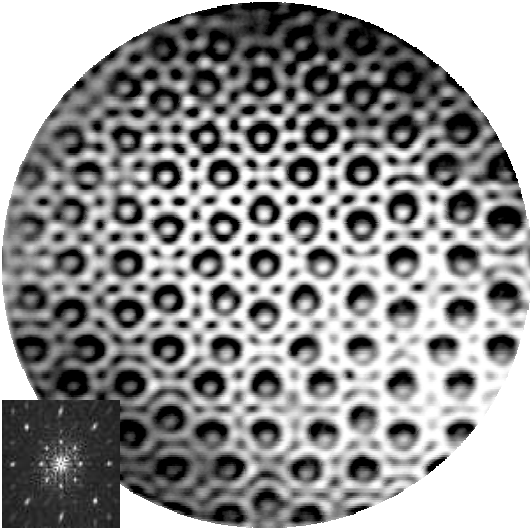


FIG. 1: Snapshot of the system vibrating vertically at a frequency of 35 Hz, under conditions of continuous wave excitation. The inset shows its Fourier spectrum with a well defined and discrete set of relevant components.

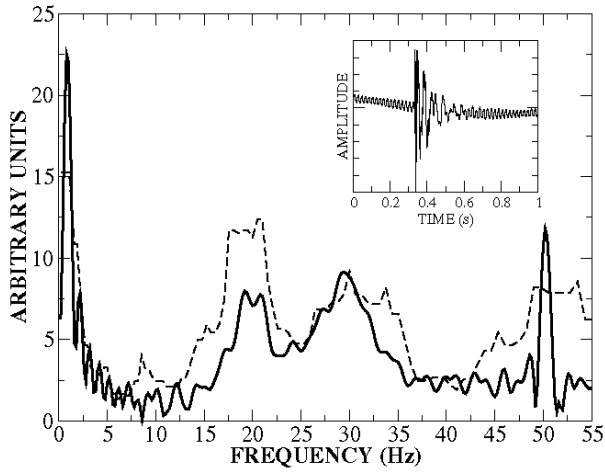


FIG. 2: The solid line is the Fourier transform of the impulsive signal as picked up by an accelerometer at the center of the vessel. It shows clear resonances at approximately 20, 30 and 50 Hz. Such resonances are specific of the system. The first peak at very low frequency, close to the origin, corresponds to the Fourier transform of the square pulse that excites the system. The signal in the time domain is presented in the inset. The dashed line is the gray scale of the subpattern along the  $\Gamma - X$  direction as represented in Fig. 4 versus frequency. The gray scale (between 0 and 1) was rescaled to match the peak of the Fourier transform at 30 Hz. Three standing waves indicating narrow band gaps appear at approximately 20, 32 and 50 Hz.

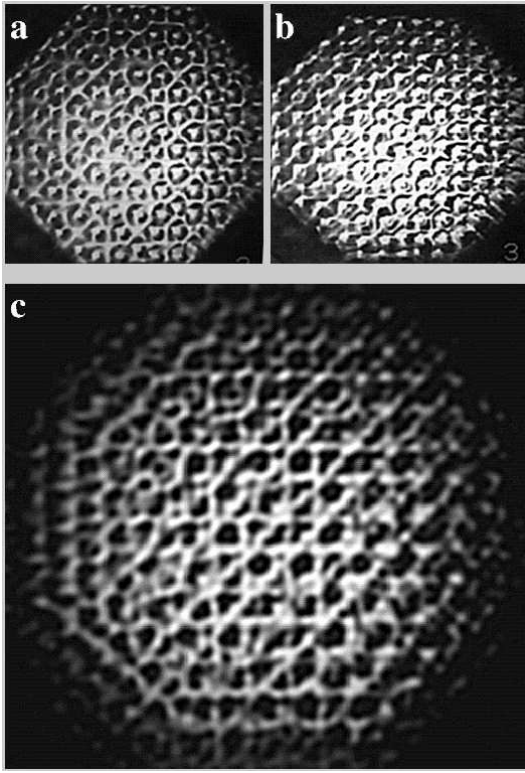


FIG. 3: Temporal sequence of patterns observed when the system is disturbed with a transverse pulse. (a) Quasicrystalline pattern observed at the start of each pulse. (b) A transitory weak turbulence is observed after 0.04 s. (c) Standing waves draw clear quasiperiodic grids between 0.08 and 0.24 s. This figure should be looked diagonally at grazing incidence.

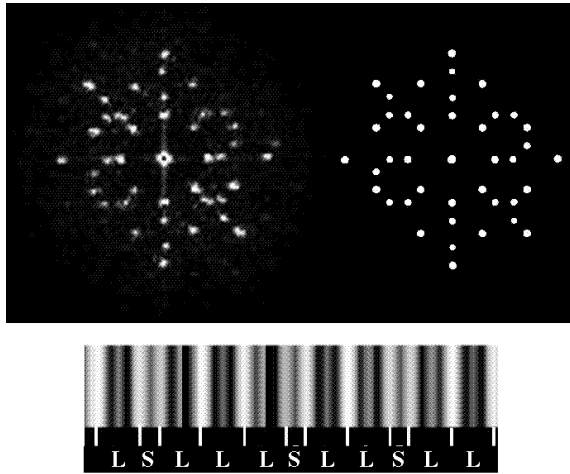


FIG. 4: Experimental (top left) and theoretical (top right) Fourier transform of the pattern shown in Fig. 3(c). At the bottom, the inverse Fourier transform of a subpattern along the  $\Gamma - X$  is shown. The Octonacci sequence is clearly recovered.

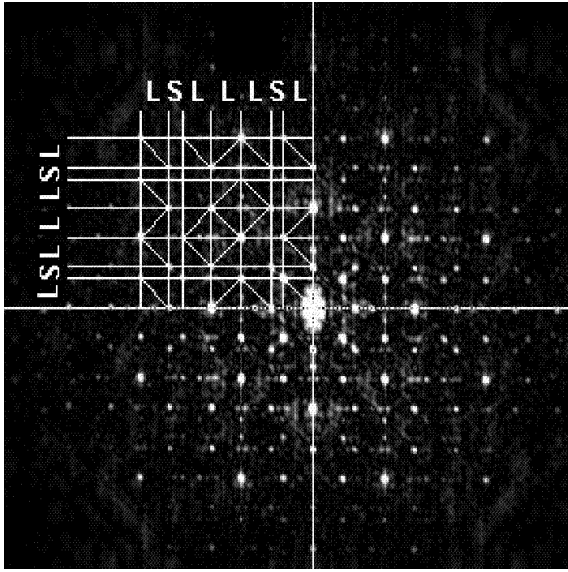


FIG. 5: Inverse Fourier transform of pattern shown in Fig. 3(b). The structure of the octagonal tiling underlying on the well quasiperiodic arrangement of the vessel bottom is recovered.